

The relativistic Bose gas in an even-dimensional space

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

1982 J. Phys. A: Math. Gen. 15 2221

(<http://iopscience.iop.org/0305-4470/15/7/031>)

View [the table of contents for this issue](#), or go to the [journal homepage](#) for more

Download details:

IP Address: 129.252.86.83

The article was downloaded on 30/05/2010 at 16:01

Please note that [terms and conditions apply](#).

The relativistic Bose gas in an even-dimensional space

Hidembergue O da Frota^{†||} and S Goulart Rosa Jr^{‡§¶}

[†] Departamento de Física, Universidade Federal do Amazonas, 69000 Manaus AM, Brasil

[‡] Instituto de Física e Química de São Carlos, Caixa Postal 369, 13560 São Carlos SP, Brasil

Received 21 December 1981

Abstract. The thermodynamic behaviour of a massive relativistic ideal Bose gas in an (even) d -dimensional space is studied. The thermodynamic functions are expressed as polynomials (or rational functions) in $k_B T/mc^2$ whose coefficients are proportional to the Bose–Einstein functions. Their closed forms permit a complete analytic discussion of the crossover from the non-relativistic limit to the ultra-relativistic limit. This transition is equivalent to the dimensional crossover from a d -dimensional to a $2d$ -dimensional non-relativistic gas behaviour.

1. Introduction

The study of the thermodynamic properties of the relativistic Bose gas has drawn much attention lately (Haber and Weldon 1981, Aragão de Carvalho and Goulart Rosa 1980a, b, Beckmann *et al* 1979, 1980, Landsberg 1981). This interest is partly derived from the study of some recent problems where the physics of the ideal relativistic quantum gas is believed to play an important role in their understanding. At the beginning of the last decade (Politzer 1973, Gross and Wilczek 1973) it was verified that non-Abelian gauge theories in four dimensions are asymptotically free. As the momentum space cut-off goes to infinity, the coupling constant vanishes resulting in a gauge field theory which is free at very short distances. It was also conjectured that the quarks would be confined in such theories. The asymptotic freedom and the quark confinement stand out as major results of the modern theory of strong interactions. More recently Kuzmin and Shaposhnikov (1979) discussed the cosmological consequences of the existence of a primordial massive photon gas therefore bringing further attention to this matter. On the other hand even though the ideal relativistic Bose gas has been an object of investigation for a long time (see Landsberg and Dunning–Davies (1965) for a historical note on the development of the subject), there are still some important aspects of its behaviour which are not yet fully understood. For instance the corrections to the ultra-relativistic limit of the $3d$ relativistic Bose gas in the low-temperature phase are unknown at the present (Aragão de Carvalho and Goulart Rosa 1980b). Another reason for this continued interest is that, besides giving a correct qualitative account of many important phenomena, the ideal Bose gas is one of the few systems which can undergo a phase transition—the Bose–Einstein condensation—and whose properties can be calculated exactly.

^{||} Bolsista de doutoramento da CAPES.

[§] Bolsista do CNPq.

[¶] Reprint requests.

In this paper we discuss the relativistic Bose gas in d -dimensional space. The qualitative difference in the analytic expressions of the thermodynamic properties resulting from the parity of the dimensionality of the gas is pointed out and we explore the simplicity which emerges from the description of the relativistic gas in an even-dimensional space. Apart from the simplicity we must add that the study of the system in an even-dimensional space is essential to the full comprehension of its criticality. It is well known that the dimensions two and four are very special ones in the study of the Bose gas. For the massive gas the Bose–Einstein condensation will occur only if $d > 2$; on the other hand for the massless gas the lower critical dimensionality is one. Also the critical exponents of the non-relativistic gas will become independent of the system dimensionality if $d > 4$; for the photon gas this will happen for $d > 2$.

In our formulation the fugacity series expansions for the thermodynamic functions are summed up and expressed as polynomials (or ratios of polynomials) of order ‘ d ’ in the parameter $(\beta mc^2)^{-1}$. The coefficients in the polynomial are, up to constants, the familiar Bose–Einstein functions. In these expressions the highest-order terms (d) give the ultra-relativistic limit while the lowest-order terms ($d/2$) are the non-relativistic contributions. The expressions for the two-dimensional gas have the simplest forms since the thermodynamical potential contains only the contributions from these extreme regimes. A d -dimensional quantum gas, the so-called Landsberg gas, defined by the approximate density of states consisting only of the ultra-relativistic and non-relativistic limits of the exact density of states, has been proposed by Landsberg (1981) and studied by Dunning–Davies (1981a). The expressions obtained for the thermodynamic functions of the Landsberg gas will coincide with the corresponding expressions for the exact system in the two-dimensional case, in other words, the Landsberg gas in an exact relativistic Bose gas in two dimensions. In higher dimensions, the exact d (even)-dimensional relativistic gas will have $d/2 - 1$ (intermediary) contributions to the grand potential besides the ultra-relativistic and non-relativistic limits. They describe the amount of mixture between the extreme regimes.

Because of the power dependence on (βmc^2) the study of the transition from the UR to the NR behaviour in the thermodynamic functions of the even-dimensional gas can be carried out analytically. As has been pointed out by May (1964) and can be seen in the comprehensive tables of thermodynamic functions of Landsberg’s (1981) paper, an ultra-relativistic gas in d dimensions behaves like a $2d$ -dimensional non-relativistic gas. Therefore the transition, which is driven by the mass, between the UR and NR limits can also be viewed as a dimensional crossover of the non-relativistic gas.

2. Formulation of the problem and the grand potential

Let us consider a gas of relativistic spinless bosons of mass m in a cubic box of volume $V = L^d$ (d will be restricted to be an even number later on). Without loss of generality we impose periodic boundary conditions on the particle wavefunctions. The energy levels are then given by

$$E_k^2 = (\hbar ck)^2 + m^2 c^4 \quad k_i = 2\pi p_i / L \quad p_i = 0, \pm 1, \dots \quad (1)$$

where $i = 1, 2, \dots, d$, and c and h are the speed of light and Planck’s constant. In

the limit of infinite volume ($L \rightarrow \infty$) the single-particle partition function

$$Z(\beta) = \sum_k \exp(-\beta E_k) \tag{2}$$

which contains all the information for a complete description of the system (Goulart Rosa and Grandy 1973, Grandy and Goulart Rosa 1981) can be evaluated by converting the sum over k to an integral containing the density of states,

$$D(E) = d \left(\frac{\pi^{1/2} L}{hc} \right)^d \frac{E(E^2 - m^2 c^4)^{(d-2)/2}}{(d/2)!} \tag{3}$$

After some straightforward algebra involving an integral representation and a recursion relationship for the modified Bessel function of the second kind $K_\nu(t)$ (Gradshteyn and Ryzhik 1965) we obtain

$$Z(\beta) = L^d \frac{2^{(d+1)/2}}{\lambda_c^d} \left(\frac{\pi}{u} \right)^{(d-1)/2} K_{(d+1)/2}(u) \tag{4}$$

where $\lambda_c = h/mc$ is the particle Compton wavelength and $u = \beta mc^2$ is the expansion parameter of the theory. We remark that the ground-state contribution to $Z(\beta)$ is lost in equation (4) due to the replacement of the summation over k by the integration over the energy. This situation is, however, easily repaired by adding the ground-state term $\exp(-u)$ directly to equation (4).

It is well known that the thermodynamic behaviour of the system can be obtained from the logarithm of the grand partition function Ξ , which has been shown to be the following integral transform of $Z(\beta)$ (Goulart Rosa and Grandy 1973).

$$\ln \Xi = -\frac{1}{2i} \int_{\gamma-i\infty}^{\gamma+i\infty} t^{-1} \cot(\pi t) \exp(\beta \xi t) Z(\beta t) dt, \quad 0 < \gamma < 1. \tag{5}$$

The integral resulting from the substitution of equation (4) into equation (5) can be evaluated employing Cauchy's theorem and a limit process. This is implemented by evaluating the integral around a finite contour consisting of a finite straight line segment of length $2R$ parallel to the imaginary axis and at a distance γ from the origin, which is closed to the right by a semi-circle of radius R . The radius is chosen such that the contour does not pass through any of the simple poles of the integrand which are located at $t = r, r = 1, 2, 3, \dots$ due to the factor $\cot(\pi t)$ of the integrand in equation (5). The original integral is obtained in the limit $R \rightarrow \infty$ imposing the condition that $\xi \leq mc^2$ in order that the contribution from the semi-circle vanishes. In order to discuss the Bose-Einstein condensation and the system's low-temperature phase, one has to set the chemical potential ξ equal to the particle rest energy. A rigorous proof that the Mellin transform formalism we are using describes this case correctly has been provided by Dunning-Davies (1981b). Recalling that the modified Bessel functions $K_\nu(t)$ are regular functions throughout the t -plane cut along the negative axis we get that, in an even-dimensional space,

$$\ln \Xi = -\ln\{1 - \exp[-\beta(mc^2 - \xi)]\} + (4\pi)^D \left(\frac{L}{\lambda_c} \right)^d \sum_{n=0}^D \frac{(D+n)!(2u)^{-(D+n)}}{n!(D-n)!} F_{D+n+1}(\alpha) \tag{6}$$

where the first term is the ground-state contribution, $2D = d$,

$$\alpha = \beta(mc^2 - \xi)$$

$$F_\sigma(\alpha) = \sum_{n=1}^{\infty} n^{-\sigma} \exp(-n\alpha) \tag{7a}$$

$$F_\sigma(0) = \zeta(\sigma). \quad (7b)$$

To obtain equation (6) we have used the fact that the modified Bessel functions of half-integer order are given by (Gradshteyn and Ryzhik 1965)

$$K_{D+1/2}(u) = \left(\frac{\pi}{2u}\right)^{1/2} \exp(-u) \sum_{n=0}^D \frac{(D+n)!}{n!(D-n)!(2u)^n}. \quad (8)$$

Although equivalent to the fugacity series expansion obtained by Beckmann *et al* (1979) and to the integral expression of Landsberg (1981), the expression for $\ln \Xi$ is written in a much simpler form which allows much easier manipulation. In the present formulation the non-relativistic ($u \rightarrow \infty$) and ultra-relativistic ($u \rightarrow 0$) limits are obtained by keeping only the first or the last term in the polynomial in equation (6), respectively. Also all the numerical studies of the intermediary case of finite u carried out by both Beckmann *et al* (1980) and Landsberg (1981) can now be done analytically.

3. Thermodynamic functions

Knowledge of $\ln \Xi$ is sufficient for the derivation of all the functions which describe the thermodynamic behaviour of the system

$$\Omega = -PV = -k_B T \ln \Xi \quad (9)$$

$$S = -\left(\frac{\partial \Omega}{\partial T}\right)_{V,\xi} = (4\pi)^D k_B \left(\frac{L}{\lambda_c}\right)^d \sum_{n=0}^D \frac{(D+n)!}{n!(D-n)!} \\ \times \left[\alpha \left(\frac{1}{2u}\right)^{D+n} F_{D+n}(\alpha) + (D+n+1) \left(\frac{1}{2u}\right)^{D+n} F_{D+n+1}(\alpha) \right] \quad (10)$$

$$P = -\left(\frac{\partial \Omega}{\partial V}\right)_{T,\xi} = (4\pi)^D \frac{k_B T}{\lambda_c^d} \sum_{n=0}^D \frac{(D+n)!}{n!(D-n)!} \left(\frac{1}{2u}\right)^{D+n} F_{D+n+1}(\alpha) \quad (11)$$

$$N = -\left(\frac{\partial \Omega}{\partial \xi}\right)_{T,V} = (4\pi)^D \left(\frac{L}{\lambda_c}\right)^d \sum_{n=0}^D \frac{(D+n)!}{n!(D-n)!} \left(\frac{1}{2u}\right)^{D+n} F_{D+n}(\alpha) \quad (12)$$

$$U = E - Nmc^2 = (4\pi)^D k_B T \left(\frac{L}{\lambda_c}\right)^d \sum_{n=0}^D \frac{(D+n)!(D+n)}{n!(D-n)!} \left(\frac{1}{2u}\right)^{D+n} F_{D+n+1}(\alpha) \quad (13)$$

where only the contributions from the excited states are written down. The critical temperature T_c is obtained from equation (12) by setting $\alpha = 0$.

From the internal energy U we can calculate the heat capacity at constant volume $C_v = (\partial U / \partial T)_{N,V}$ and its jump Δ at the critical temperature

$$C_v^- = C_v(T \leq T_c) = (4\pi)^D \left(\frac{L}{\lambda_c}\right)^d k_B \sum_{n=0}^D \frac{(D+n)!(D+n)(D+n+1)}{n!(D-n)!} \left(\frac{1}{2u}\right)^{D+n} F_{D+n+1}(0) \quad (14a)$$

$$\begin{aligned}
 C_v^+ &= C_v(T > T_c) \\
 &= (4\pi)^D \left(\frac{L}{\lambda_c}\right)^d k_B \left[\sum_{n=0}^D \frac{(D+n)!(D+n)(D+n+1)}{n!(D-n)!} \left(\frac{1}{2u}\right)^{D+n} F_{D+n+1}(\alpha) \right. \\
 &\quad - \sum_{n=0}^D \frac{(D+n)!(D+n)}{n!(D-n)!} \left(\frac{1}{2u}\right)^{D+n} F_{D+n}(\alpha) \\
 &\quad \times \left(\sum_{m=0}^D \frac{(D+m)!(D+m)}{m!(D-m)!} \left(\frac{1}{2u}\right)^{D+m} F_{D+m}(\alpha) \right) \\
 &\quad \left. \times \left(\sum_{m=0}^D \frac{(D+m)!}{m!(D-m)!} \left(\frac{1}{2u}\right)^{D+m} F_{D+m-1}(\alpha) \right)^{-1} \right] \tag{14b}
 \end{aligned}$$

$$\begin{aligned}
 \frac{C_v^- - C_v^+}{Nk_B} &= \left[\sum_{n=0}^D \frac{(D+n)!(D+n)}{n!(D-n)!} \left(\frac{1}{2u_c}\right)^{D+n} F_{D+n}(0) \right]^2 \\
 &\quad \times \left[\sum_{n=0}^D \frac{(D+n)!}{n!(D-n)!} \left(\frac{1}{2u_c}\right)^{D+n} F_{D+n-1}(0) \right] \\
 &\quad \times \left[\sum_{n=0}^D \frac{(D+n)!}{n!(D-n)!} \left(\frac{1}{2u_c}\right)^{D+n} F_{D+n}(0) \right]^{-1} \tag{15}
 \end{aligned}$$

where $u_c = mc^2/k_B T_c$.

Noticing that $F_\sigma(0)$ is infinite for $\sigma \leq 1$ we see immediately from equations (12) and (15) that for a massive gas there is Bose-Einstein condensation for (even) dimensions $d \geq 4$ but a gap in the heat capacity will develop only at the (even) dimensions $d \geq 6$, since for $d = 4$ we have $\Delta = 0$. For the massless gas there is Bose-Einstein condensation at the (even) dimensions $d \geq 2$ and as the only term in the denominator of equation (15) is $n = D$ the heat capacity will display a jump whenever (even) $d \geq 4$. A numerical study of Δ/Nk_B as function of d (parametrised by βmc^2) has been carried out by Landsberg (1981), Dunning-Davies (1981) and Beckmann *et al* (1980). The latter authors have also obtained Δ numerically as a function of βmc^2 for (both even and odd) dimensions between five and ten. In these works the jump expression is given in terms of integrals which do not admit analytic solution for the massive gas and a numerical evaluation of these integrals is necessary. Again this must be compared with the simple rational expression obtained above.

4. Conclusions

We have used the grand canonical formalism to investigate the effects of dimensionality on the thermodynamic functions of an ideal massive relativistic Bose gas. All expressions derived are exact and their simple functional forms result from the truncated, but correct, dependence on β of the single-particle partition function in an even-dimensional space. In the present formulation, several pieces of information are quickly obtained directly from the equation of the logarithm of the grand partition function. The non-relativistic as well as the ultra-relativistic limits are always present in the expressions for the thermodynamic functions, the other terms representing their admixture. The statement (Aragão de Carvalho and Goulart Rosa 1980b) that the thermodynamic behaviour of the massive gas in the limit $T \rightarrow 0$ is dominated by the non-relativistic contributions is now transparent in the present formulation. Below

the critical temperature ($\alpha = 0$) the calculation and comparison of each contribution in the polynomial expressions of the thermodynamic functions are easily carried out, allowing us to estimate the (mass/temperature) range where the non-relativistic or ultra-relativistic contributions are dominant. This situation must be contrasted with the fact that the corrections to the ultra-relativistic term in the low-temperature phase of a $3d$ gas are not known at present, as pointed out in the introduction. Since the evaluation and the manipulation of the thermodynamic functions can be carried out more easily for even-dimensional systems, we can use the results obtained for two consecutive even-dimensional systems as an estimation, or better as bounds, for the corresponding quantities of the interpolated odd-dimensional system. This is an alternative approach to the use of an approximate density of states. As a final comment we mention that a similar qualitative difference in the thermodynamic functions of a relativistic Fermi gas will also result from the parity of the dimensionality of the system.

References

- Aragão de Carvalho C A and Goulart Rosa S Jr 1980a *J. Phys. A: Math. Gen.* **13** 989
— 1980b *J. Phys. A: Math. Gen.* **13** 3233
Beckmann R, Karsch F and Miller D E 1979 *Phys. Rev. Lett.* **43** 1277
— 1980 *preprint BI-TP 80/19 August*
Dunning-Davies J 1981a *J. Phys. A: Math. Gen.* **14** 3005
— 1981b *J. Phys. A: Math. Gen.* **14** 3013
Goulart Rosa S Jr and Grandy W T Jr 1973 *Rev. Bras. Fis.* **3** 537
Gradshteyn I S and Ryzhik I M 1965 in *Tables of Integrals, Series and Products* (New York: Academic)
Grandy W T Jr and Goulart Rosa S Jr 1981 *Am. J. Phys.* **49** 570
Gross D J and Wilczek 1973 *Phys. Rev. Lett.* **30** 1343
Haber H E and Weldon H A 1981 *Phys. Rev. Lett.* **46** 1497
Kuzmin V A and Shaposhnikov M E 1979 *Phys. Rev. Lett.* **69A** 462
Landsberg P T 1981 in *Statistical Mechanics of Quarks and Hadrons* ed H Satz (Amsterdam: North-Holland)
Landsberg P T and Dunning-Davies J 1965 in *Statistical Mechanics of Equilibrium and Non-equilibrium* ed J Meixner (Amsterdam: North-Holland)
May R M 1964 *Phys. Rev.* **135** A1515
Politzer H D 1973 *Phys. Rev. Lett.* **30** 1346